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$SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$, the classical and quantum q-deformations of the SU(2) algebra

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Abstract. By means of classical harmonic oscillators and their canonical quantization, we show that the q-deformation of the SU(2) Lie algebra can be realized at a classical level in the sense of Poisson brackets, denoted by $SU_{q,\hbar\to 0}(2)$, and that the canonical quantization of the system leads to the quantum q-deformation of the SU(2) algebra, denoted by $SU_{q,\hbar}(2)$. This means that the q-deformation and the $\hbar$-quantization are in general two different concepts.

1. Introduction

Recently there has been a great deal of interest in the study of the quantum groups or the quantized enveloping algebras [1–3]. The nomenclature of these beautiful algebraic structures reflects the fact that the q-deformations have so far always emerged together with quantization characterized by the Planck constant $\hbar$ or by some parameter playing a role loosely like $\hbar$. In the classical limit $\hbar \to 0$, the q-deformation of the Lie algebra disappears and the algebraic structure reduces to the usual Lie algebraic structure. We find, however, that this is not the case in general [4]. In fact, the q-deformation of Lie algebras can be realized at a classical level in the sense of Poisson brackets, while after quantization of the system these classical q-deformed structures give rise to their quantum counterparts. Namely, the q-deformation and the $\hbar$-quantization are different concepts in principle.

In this paper, we present a concrete example, the case of the SU(2) algebra, by means of harmonic oscillators. First, we realize the classical q-deformation of the SU(2) algebra, denoted by $SU_{q,\hbar\to 0}(2)$, by a set of Poisson brackets of q-deformed generators $J'_\pm$ and J'_3 on the phase space V with symplectic form Ω of two undeformed one-dimensional linear oscillators. In the limit $q \to 1$, we find that $SU_{q,\hbar\to 0}(2)$ reduces to the usual SU(2) algebra in the sense of Poisson brackets of the undeformed generator $J_\pm$, J_3. It is of interest that in terms of complex coordinates in the phase space the q-deformation is then associated with Beltrami deformations with certain dilatation ratio. Second, we quantize the system canonically and recover the realization of quantum q-deformation of the SU(2) algebra, denoted by $SU_{q,\hbar}(2)$ in our notation, in terms of the q-analogues of the quantum oscillators [5–8]. In our approach, it is easy to see that the classical counterpart of $SU_{q,\hbar}(2)$ is just $SU_{q,\hbar\to 0}(2)$ and the undeformed limit, i.e. $q \to 1$, of $SU_{q,\hbar}(2)$ is nothing but the realization of the SU(2) algebra in terms of undeformed quantum oscillators.

2. Classical q-deformed SU(2) algebra, $SU_{q,\hbar\to 0}(2)$

Let us consider a system of two linear oscillators with same mass m and frequency ω_0. In terms of complex coordinates the Hamiltonian of the system reads

$$H = z_1\bar{z}_1 + z_2\bar{z}_2. \tag{1}$$

Here, $z_i = (p_i + \mathrm{i}q_i)/\sqrt{2}$ $i = 1, 2$, and we take $m = w_0 = 1$. The symplectic form Ω is given by

$$\Omega = -\mathrm{i}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2)\,. \tag{2}$$

The phase space with the symplectic form (2) is denoted by (V, Ω), which is a two-dimensional complex symplectic space.

Using the basic Poisson brackets

$$\{\bar{z}_i, z_i\} = -\mathrm{i} \qquad i = 1, 2, \qquad \text{(others vanish).} \tag{3}$$

It is easy to show that

$$J_+ = z_1\bar{z}_2 \qquad J_- = z_2\bar{z}_1 \qquad J_3 = \tfrac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2) \tag{4}$$

form an SU(2) algebra in the sense of Poisson brackets

$$\{J_3, J_\pm\} = -\mathrm{i}\,(\pm J_\pm) \qquad \{J_+, J_-\} = -\mathrm{i}\,2J_3. \tag{5}$$

While their Hamiltonian vector fields $X_{J_\pm}, X_{J_3}$ form an SU(2) algebra with Lie brackets.

Now Let us construct a classical q-deformed oscillator system described by a q-deformed Hamiltonian,

$$H' = z'_1\bar{z}'_1 + z'_2\bar{z}'_2 \tag{6}$$

on the symplectic space (V, Ω), in which

$$z'_i = z'_i(z_i, \bar{z}_i) \qquad \bar{z}'_i = \bar{z}'_i(z_i, \bar{z}_i) \qquad i = 1, 2 \tag{7}$$

such that a set of deformed observables $J'_\pm, J'_3 = J_3$ form a classical q-deformed SU(2) algebra $SU_{q,\hbar\to 0}(2)$ in the sense of Poisson brackets defined by the symplectic form Ω in (2):

$$J'_+ = z'_1\bar{z}'_2 \qquad J'_- = z'_2\bar{z}'_1 \qquad J'_3 = J_3 \tag{8}$$

$$\{J'_3, J'_\pm\} = \mp\mathrm{i}\,J'_\pm \tag{9}$$

$$\{J'_+, J'_-\} = -\mathrm{i}\,[2J_3] \tag{10}$$

where†

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh \gamma x}{\sinh \gamma} \qquad \gamma = \log q.$$

It is easy to prove that provided z_i', $\bar{z}_i'$ $(i = 1,2)$ satisfy

$$z_i'\bar{z}_i' = \frac{\sinh \gamma z_i \bar{z}_i}{(\gamma \sinh \gamma)^{1/2}} \tag{11}$$

$$\{z_i', \bar{z}_i'\} = -\mathrm{i}\left(\frac{\gamma}{\sinh \gamma}\right)^{1/2} \cosh \gamma z_i \bar{z}_i \tag{12}$$

the q-deformed algebraic relations (9) and (10) hold. Also (11) and (12) have a set of solutions

$$z_i' = (\gamma \sinh \gamma)^{-1/4}\left(\frac{\sinh \gamma z_i \bar{z}_i}{z_i \bar{z}_i}\right)^{1/2} z_i \mathrm{e}^{\mathrm{i}\gamma \alpha_i(z_i \bar{z}_i)} \tag{13}$$

$$\bar{z}_i' = (\gamma \sinh \gamma)^{-1/4}\left(\frac{\sinh \gamma z_i \bar{z}_i}{z_i \bar{z}_i}\right)^{1/2} \bar{z}_i \mathrm{e}^{-\mathrm{i}\gamma \alpha_i(z_i \bar{z}_i)} \tag{14}$$

where $\alpha_i = \alpha_i(z_i\bar{z}_i)$ $i = 1,2$, are arbitrary real functions of $z_i\bar{z}_i$.

The Hamiltonian vector fields $X_{J'_\pm}$ and X_{J_3} also form an algebra

$$[X_{J_3'}, X_{J'_\pm}] = \pm X_{J'_\pm} \tag{15}$$

$$[X_{J'_+}, X_{J'_-}] = -\mathrm{i}\, X_{[2J_3]}. \tag{16}$$

This is also a deformed SU(2) algebra but it is not isomorphic to $\mathrm{SU}_{q,h\to 0}(2)$ given by (9) and (10).

In terms of the canonical coordinates q_i and canonical momenta p_i, the Hamiltonian of q-deformed oscillators H' can be written as

$$H' = \sum_{i=1,2} (\gamma \sinh \gamma)^{-\frac{1}{2}} \sinh\left(\frac{\gamma}{2}(p_i^2 + q_i^2)\right) \tag{17}$$

The canonical equations of motion

$$\dot{p}_i = \{p_i, H'\} \tag{18}$$

$$\dot{q}_i = \{q_i, H'\} \tag{19}$$

can be solved exactly

$$p_i = A_i \cos\left[\left(\frac{\gamma}{\sinh \gamma}\right)^{1/2} \cosh\, \gamma \frac{A_i^2}{2}\right] t \tag{20}$$

$$q_i = A_i \sin\left[\left(\frac{\gamma}{\sinh \gamma}\right)^{-1/2} \cosh\, \gamma \frac{A_i^2}{2}\right] t. \tag{21}$$

† In this paper we take γ to be real; that is for the case where q is not a root of unity.

This means that the q-deformed oscillators are still harmonic oscillators although their frequencies

$$\omega_i' = \left[\left(\frac{\gamma}{\sinh\gamma}\right)^{1/2} \cosh \gamma \frac{A_i^2}{2}\right]\omega_0 \qquad \omega_0 = 1 \tag{22}$$

are dependent on their amplitudes A_i.

It is worthwhile pointing out that from the geometric point of view the deformations (13) and (14) are Beltrami deformations, i.e. quasi-conformal deformations (see, for example, [9]), since we can define the Beltrami coefficients on V

$$\mu_i(z_i, \bar{z}_i) = \frac{\bar{\partial} z_i'}{\partial z_i'} \qquad i = 1, 2 \tag{23}$$

which satisfy

$$|\mu_i(z_i, \bar{z}_i)| < 1 \qquad i = 1, 2. \tag{24}$$

This may shed new light on the geometrical meaning of the classical q-deformation of Lie algebras. We leave this subject for further investigation.

3. Quantum q-deformed SU(2) algebra, $SU_{q,\hbar}(2)$

Let us now quantize canonically the q-deformed oscillators and show the quantum q-deformed SU(2) algebra, $SU_{q,\hbar}(2)$. Since the q-deformed oscillator system has been given on the phase (V, Ω) whereupon the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ with Poisson brackets has been realized, while the phase space (V, Ω) has not been changed, we may directly carry out the canonical quantization with taking care of the ordering of the operators as usual.

Under the canonical quantization, the basic Poisson brackets (3) become†

$$[a_i, a_i^+] = \delta_{ij} \qquad i = 1, 2 \qquad \text{(others vanish)}. \tag{25}$$

Then the Poisson algebra SU(2) in (5) becomes

$$[J_3, J_\pm] = \pm J_\pm \tag{26}$$

$$[J_+, J_-] = 2J_3 \tag{27}$$

with

$$J_+ = a_1^+ a_2 \qquad J_- = a_2^+ a_1 \qquad J_3 = \tfrac{1}{2}(a_1^+ a_1 - a_2^+ a_2). \tag{28}$$

Having the Poisson algebraic relations (9) and (10) for the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$, the expressions (8) for its generators and the quantum canonical commutation relations (25), the quantum q-deformed SU(2) algebra, $SU_{q,\hbar}(2)$, can be realized more or less straightforwardly.

† We use the notation correspondence between $(z_i, \bar{z}_i; z_i', \bar{z}_i')$ and their quantum counterparts $(a_i^+, a_i; a_i'^+, a_i')$ but the same notations for others. We also take $\hbar = 1$ for the sake of convenience.

The quantum versions of relations (13) and (14) read

$$a_i'^{+} = (\gamma^{-1}\sinh\gamma)^{-1/4}\left(\frac{[N_i]}{N_i}\right)^{1/2} e^{i\gamma\alpha_i(N_i)} a_i^{+} \tag{29}$$

$$a_i' = a_i(\gamma^{-1}\sinh\gamma)^{-1/4}\left(\frac{[N_i]}{N_i}\right)^{1/2} e^{-i\gamma\alpha_i(N_i)} \tag{30}$$

where $N_i = a_i^{+}a_i$, the quantum operator number of the ith undeformed oscillator. $a_i'^{+}$ and a_i' satisfy

$$a_i'^{+}a_i' = \left(\frac{\sinh\gamma}{\gamma}\right)^{1/2}[N_i] \tag{31}$$

$$[a_i', a_j'^{+}] = \left(\frac{\sinh\gamma}{\gamma}\right)^{1/2}([N_i+1]-[N_i])\delta_{ij} \tag{32}$$

which are the quantum counterparts of (11)and (12). The quantum versions of the generators (8) are naturally

$$J_+' = a_1'^{+}a_2' \qquad J_-' = a_2'^{+}a_1' \qquad J_3' = J_3 = \tfrac{1}{2}(a_1^{+}a_1 - a_2^{+}a_2). \tag{33}$$

They obey the following relations

$$[J_3', J_\pm'] = \pm J_\pm' \tag{34}$$

$$[J_+', J_-'] = \frac{\sinh\gamma}{\gamma}[2J_3']. \tag{35}$$

These are just the required algebraic relations for $SU_{q,\hbar}(2)$, the quantum q-deformed SU(2) algebra. Notice that the constant factors, the roots of $\sinh\gamma/\gamma$, in(31)–(35) are irrelevant. They may disappear by shifting the deformed operators with corresponding scales. In this sense, the relations (31) and (32) are the same as that given in [5-8]. But the relations (29) and (30) are different from the ones given in [8] by phase factors.

4. Remarks

We conclude with some remarks. First, we have seen that the classical q-deformed SU(2) symmetry, $SU_{q,\hbar\to 0}(2)$, and its quantum version, $SU_{q,\hbar}(2)$, have been realized by a Hamiltonian system of (q-deformed) nonlinear harmonic oscillators and its canonical quantization, respectively. As was mentioned at the beginning, this means that the q-deformations of Lie algebras and the $\hbar$-quantization are two independent concepts in general. Second, we have not touched, in this paper, the representation theory of $SU_{q,\hbar}(2)$. This is of course an important subject. The physical implications of both classical and quantum q-deformed observables are also of interest, especially in the case of q being a root of unity. Third, we have not set up the wonderful structures of 'quantum groups' [1,2]. It seems, however, to be straightforward to build such structures, like the Hopf algebra relation etc, in the quantum harmonic oscillator approach. Finally, our approach can be generalized naturally to deal with the classical and quantum q-deformations of Lie algebras of all kinds. All these and other relevant subjects are under investigation.

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